ON THE SOLUTION OF CERTAIN PROBLEMS WITH AN UNKNOWN BOUNDARY IN THE THEORY OF ELASTICITY AND PLASTICITY

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PMM Vol.28, № 1, 1964, pp.141-145

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(Received September 30, 1963)

In previous papers by the author [1] and [2], solutions were obtained for certain elastic-plastic problems and for problems on the local buckling of a membrane. It was found that these solutions hold only up to a certain value of the load parameter (at which a cusp appears on the unknown boundary). It should be noted that solutions were sought in a class of functions (stresses) which were bounded everywhere in the elastic region, including the unknown boundary (here we do not take into account the inevitable singularities of the functions caused by the presence of a cusp on the known boundary, as is the case, for example, in the problem of buckling of a membrane).

In what follows we shall derive solutions to the two problems indicated in a class of functions (stresses) which are unbounded at certain points on the unknown boundary corresponding to the cusps in the first solution. The solutions obtained are a sequel to those derived in references [1] and [2] for large values of the load parameter and coincide with them only for one value of the load parameter. Above this value the solutions given in papers [1] and [2] no longer hold. According to the solutions found, there always exists a cusp on the contour of the unknown boundary.

Two examples will be used to illustrate a sufficiently general method by which the two solutions can be combined to form a unique solution valid over the whole range of parameters.

1. The elastic-plastic problem for a plate with a circular orifice.

1) Consider the elastic-plastic problem for an infinite plate with a circular orifice of radius R under a uniform state of stress

$$\sigma_x = \sigma_x^{\infty}, \qquad \sigma_y = \sigma^{\infty}, \qquad \tau_{xy} = 0$$

On the contour of the orifice there is applied a constant normal stress $\sigma_r = p$ with the tangential stress $\tau_{r\theta}$ zero (in polar coordinates r and θ). As a plasticity condition in the plastic region we accept the Tresca – St. Venant condition. We assume that $0 \leqslant p \leqslant \sigma_s$, where σ_s is the plasticity constant. The formulation of the problem and the notation are the same as in [2].

In the parametric plane of ζ the boundary-value problem may be written in the form [2]

$$4\,\mathrm{Re}\,\varphi(\zeta) = 2\sigma_s + \frac{R(p-\sigma_s)}{|\omega(\zeta)|} \qquad \text{for } |\zeta| = 1$$

$$\frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\varphi'(\zeta) + \psi(\zeta) = \frac{R(\sigma_s - p)\,\overline{\omega(\zeta)}}{2\omega(\zeta)\,|\omega(\zeta)|} \qquad \text{for } |\zeta| = 1$$

$$\varphi(\zeta) = {}^1/_4(\sigma_x^\infty + \sigma_y^\infty) + O(\zeta^{-2}), \qquad \omega(\zeta) = O(\zeta)$$

$$\psi(\zeta) = {}^1/_2(\sigma_y^\infty - \sigma_x^\infty) + O(\zeta^{-2}), \qquad \text{for } \zeta \to \infty$$

Consider functional Equation

$$\frac{\varphi'(\zeta)}{\omega'(\zeta)}\,\overline{\omega}\left(\frac{1}{\zeta}\right) + \psi(\zeta) = \frac{R(\sigma_s - p)\,\overline{\omega}(1/\zeta)}{2\omega(\zeta)\sqrt{\omega(\zeta)\,\overline{\omega}(1/\zeta)}}$$

From formula (3.13) in [2] the solution to this equation is

$$\omega(\zeta) = \frac{c_0}{\zeta^3}\left(\zeta^2 + \frac{c_1}{2c_0}\right)^2, \qquad c_1^2 = 4c_0c_3 \tag{1.1}$$

$$\psi(\zeta) = \frac{R(\sigma_s - p)\sqrt{c_3}\,\zeta^4(\zeta^2 + c_1/2c_3)}{2c_0^{3/2}(\zeta^2 + c_1/2c_0)^3} - \frac{\varphi'(\zeta)}{\omega'(\zeta)}\,\overline{\omega}\left(\frac{1}{\zeta}\right)$$

where $\varphi(\zeta)$ is given by the solution to Dirichlet's problem (3.14) for the exterior of the circle $|\zeta| > 1$

$$2\,\mathrm{Re}\,\varphi(\zeta) = \sigma_s + \frac{R(p-\sigma_s)\sqrt{c_0c_3}}{c_1(c_3 - c_0)}\,\zeta^2\left(\frac{1}{\zeta^2 + c_1/2c_3} - \frac{1}{\zeta^2 + c_1/2c_0}\right) \quad \text{for } |\zeta| = 1 \tag{1.2}$$

In order to be precise we shall assume always that $\sigma_y^\infty \geqslant \sigma_x^\infty \geqslant 0$, and we shall seek a solution to Dirichlet's boundary-value problem (1.2) in a class of functions which have poles at the points $\zeta = \pm 1$. Since there is no concentrated force at corresponding points on the unknown contour of the physical plane of z it follows that at the points $\zeta = \pm 1$ condition

$$\omega'(\pm 1) = 0 \tag{1.3}$$

must be satisfied.

This condition means that the as yet unknown dividing line between the elastic and plastic regions always has cusps at points corresponding to $\zeta = \pm 1$.

The stresses in the elastic region in the neighborhood of a cusp have a singularity of the type $1/\sqrt{z}$. Taking into account Formula (1.1) we find from condition (1.3) that

$$2c_0 = 3c_1 \tag{1.4}$$

It can be shown that the solution to Dirichlet's problem (1.2) in the class of functions indicated, which at present does not satisfy the condition at infinity, is of the form

$$\varphi(\zeta) = A\,\frac{\zeta^2 + 1}{\zeta^2 - 1} - \frac{3R(p-\sigma_s)}{16c_0}\,\frac{1}{\zeta^2 + {}^1/_3} + \frac{1}{2}\sigma_s + \frac{9R(p-\sigma_s)}{32c_0} \tag{1.5}$$

where A is a real constant. In order to find the constants c_0 and A we have two conditions at infinity

$$\varphi(\zeta) \to {}^1/_4(\sigma_x^\infty + \sigma_y^\infty), \qquad \psi(\zeta) \to {}^1/_2(\sigma_y^\infty - \sigma_x^\infty) \qquad \text{for } \zeta \to \infty \tag{1.6}$$

Making use of Formulas (1.1),(1.5) and (1.6) we obtain

$$c_0 = \frac{6R(\sigma_s - p)}{4\sigma_s + 7\sigma_y^\infty - 11\sigma_x^\infty}, \qquad A = \frac{37}{64}\sigma_y^\infty - \frac{17}{64}\sigma_x^\infty - \frac{5}{16}\sigma_s \tag{1.7}$$

The functions $\omega(\zeta)$ and $\psi(\zeta)$ may be written in the form

$$\omega(\zeta)=\frac{c_0}{\zeta^3}\left(\zeta^2+\frac{1}{3}\right)^2,\qquad \psi(\zeta)=\frac{R(\sigma_s-p)\,\zeta^4(\zeta^2+3)}{6c_0(\zeta^2+{}^1/_3)^3}-\frac{\varphi'(\zeta)}{\omega'(\zeta)}\,\overline{\omega}\left(\frac{1}{\zeta}\right) \quad (1.8)$$

We can write the equation of the contour separating the elastic and plastic regions in the parametric form

$$x(t)={}^1/_3c_0\,(5\cos t+{}^1/_3\cos 3t)$$
$$y(t)={}^1/_3c_0\,(\sin t-{}^1/_3\sin 3t)$$
$$(0\leqslant t\leqslant 2\pi) \quad (1.9)$$

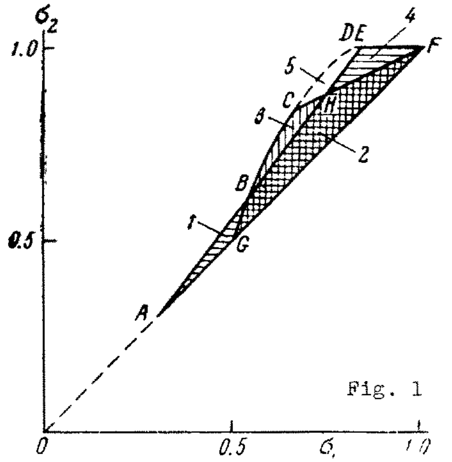

Fig. 1

We see that the dividing contour between the elastic and plastic regions for all values of the load parameters remains similar to the contour of the solution of [2] for a critical value of the parameter $\alpha={}^2/_3$; if this value is exceeded the solution of [2] no longer has any physical meaning. The solution of (1.5),(1.7) and (1.8) exists for all values of the load parameters which satisfy the condition that the plastic zone completely surrounds the circular orifice

$$4c_0\geqslant 9R \quad (1.10)$$

2. Let us now consider the question of the existence and uniqueness of a solution to the initial elastic-plastic problem. We shall first find the regions of the existence of the solution [2] and of the solution of (1.5), (1.7) and (1.8). These solutions depend on three load parameters $p^*=p/\sigma_s$, $\sigma_1=\sigma_x^\infty/\sigma_s$ and $\sigma_2=\sigma_y^\infty/\sigma_s$ which form a three-dimensional load-parameter space. For the sake of simplicity and clearness we shall set $p^*=0$, so that the region of variation of the two load parameters σ_1 and σ_2 will be the interior of a triangle with corners at the points (0,0), (0,1) and (1,1) in the $\sigma_1\sigma_2$ plane (see the figure) (in consequence of the inequality

$$\sigma_s\geqslant\sigma_y^\infty\geqslant\sigma_x^\infty\geqslant 0\,).$$

The regions of existence of the solutions is indicated in the figure as follows: existence of the solution of (1.5), (1.7) and (1.8) is denoted by horizontal hatching and the region of existence of the solution of [2] is denoted by vertical hatching. The straight line *ABHE* gives the load parameters for which the dividing line between the elastic and plastic regions of the solution of (1.5), (1.7) and (1.8) touches the circular orifice. Equation of this line has the form

$$21\sigma_2-33\sigma_1+4=0 \quad (1.11)$$

Equation of the straight line *AGF* is $\sigma_2-\sigma_1=0$. The curve *GBCD* gives the load parameters for which the contour separating the elastic and plastic regions, in accordance with the solution of [2], touches the circular orifice. Equation of this curve is

$$\left(\frac{\sigma_1+\sigma_2-1}{3-\sigma_1-\sigma_2}\right)^3+\frac{\sigma_1+\sigma_2-1}{3-\sigma_1-\sigma_2}=\frac{\sigma_2-\sigma_1}{2-\sigma_1-\sigma_2} \quad (1.12)$$

The straight line *CHF* gives the load parameters for which a cusp appears on the dividing contour between the elastic and plastic zones in the solution of [2]. Its Equation is

$$37\sigma_2-17\sigma_1-20=0 \quad (1.13)$$

It can be seen from the figure that in the regions 1, 3 and 4 the solution is unique and that in the region 2 there are two solutions. It can be shown that with an a priori assumption of the uniqueness of the solution to the functional equation, single-sheeted solutions to the initial problem in the class of functions (stresses) which are almost everywhere bounded, would no longer exist. We shall show that in this case the solutions obtained must be combined and the question of uniqueness solved.

D e f i n i t i o n . We define the solution to the elastic-plastic problem as correct if it is a continuous function of all the parameters of the problem almost everywhere within the range of variation of the independent variables x and y.

We require the solution to the initial elastic-plastic problem to be correct. The preceding two solutions are both continuous functions of the parameters everywhere within the region of existence of the solution in the σ_1, σ_2 plane. We must now check the continuity of the solutions on the boundaries of the region of existence. We immediately find that in the regions 1 and 2 the solution (1.5), (1.7) and (1.8) is incorrect, since on the straight line AGF, when $\sigma_1 = \sigma_2$, it does not transform to the familiar axisymmetric solution of the elastic-plastic problem. From this point of view the solution of [2] is correct everywhere within the regions of existence 2 and 3, since when $\sigma_1 = \sigma_2$ it transforms continuously into an axisymmetric solution. However, the question of the continuity of the solution of [2] for the elastic-plastic problem on the boundary GBC remains open, since no solution of the elastic-plastic problem is known when the plastic zone only partially surrounds the circular orifice. Let us make the a priori assumption that the solution of [2] is continuous on the boundary $GBCH$ and that the solution (1.5), (1.7) and (1.8) is continuous on the segment HE. It is easy to verify that on the segment HF the solution of (1.5), (1.7) and (1.8) coincides with the solution of [2]. From this following Theorem my be stated.

T h e o r e m 1. A solution of the elastic-plastic problem based on the above a priori assumption exists in the regions 2, 3 and 4 of the variation of the load parameters and is unique in the class of correct, single-sheeted solutions, bounded almost everywhere; in regions 2 and 3 the solution (bounded everywhere) is given by Formulas in [2] and in region 4 the solution (bounded almost everywhere) is given by Formulas (1.5), (1.7) and (1.8).

We note that the mathematical assumption on the infiniteness of stresses in the neighborhood of the cusp is explained physically by the existence of statically indeterminable plastic zones in this neighborhood.

This factor if a defect of the present solution and in this sense it is an approximate solution; however, it does at least have a definite physical meaning within the region 4 at points sufficiently close to the segment HF. Moreover, the concept of correctness of the solution and the method of combining the solutions are apparently of a completely general nature. as will be shown from the following example from a different field.

2. Local buckling of a membrane containing a slot. Suppose that an infinite membrane having a slot $(-l, +l)$ which is free from loading is subjected to a uniform field of tensile stresses as a result of which a zone of local buckling occurs in the membrane close to the slot. The general formulation of the problem of local buckling of membranes and the solution of certain specific problems are given in [1].

The boundary conditions on the unknown contour L separating the buckled and unbuckled zones may be written in the form [1]

$$\operatorname{Re}\Phi(z) = 0, \qquad \bar{z}\Phi'(z) + \Psi(z) = 0 \qquad \text{on } L$$

$$\Phi(z) = {}^1/_4(\sigma_x^\infty + \sigma_y^\infty) + O(z^{-2}), \quad \Psi(z) = {}^1/_2(\sigma_y^\infty - \sigma_x^\infty) + O(z^{-2}) \qquad \text{for } z \to \infty$$

Here we shall use the notation as in [1]. According to [1], on the exterior of a single cut along the real axis of the parametric plane of ζ we have boundary-value problem

$$\mathrm{Re}\,\varphi(\zeta) = 0, \qquad \overline{\omega(\zeta)} + \chi(\zeta) = 0 \tag{2.1}$$

$$\varphi(\zeta) = {}^1/_4\,(\sigma_x{}^\infty + \sigma_y{}^\infty) + O(\zeta^{-2}), \quad \omega(\zeta) = O(\zeta), \quad \chi(\zeta) = O(\zeta^3) \quad \text{for} \quad \zeta \to \infty$$

The function $\varphi(\zeta)$ is unbounded, and the functions $\omega(\zeta)$ and $\chi(\zeta)$ are bounded in the vicinity of the ends of the segment $\zeta = \pm 1$. We seek solution to Dirichlet's problem for the function $\varphi(\zeta)$ in a class of functions unbounded at the point $\zeta = 0$, which is the image of the point of maximum displacement in the buckled region of the membrane (and which develops into a cusp in the solution of [1], after which this solution no longer holds). We find that

$$\varphi(\zeta) = \frac{(\sigma_x{}^\infty + \sigma_y{}^\infty)\,\zeta^2 + F}{4\zeta\,\sqrt{\zeta^2 - 1}}, \qquad \sqrt{\zeta^2 - 1} = \zeta + O(\zeta^{-1}) \quad \text{for} \quad \zeta \to \infty \tag{2.2}$$

where F is real constant.

The function $\omega(\zeta)$ and $\chi(\zeta)$ are determined by Formulas in [1]

$$\begin{aligned}\omega(\zeta) &= (-A\zeta^2 - B\zeta + E)\sqrt{\zeta^2 - 1} + A\zeta^3 + B\zeta^2 + C\zeta + D\\ \chi(\zeta) &= (-A\zeta^2 - B\zeta + E)\sqrt{\zeta^2 - 1} - A\zeta^3 - B\zeta^2 - C\zeta - D\end{aligned} \tag{2.3}$$

For finding the unknown real constants A, B, C, D, E and F we have the following conditions:

1) the conditions of absence of concentrated forces at points in the z plane which are images of the points $\zeta = 0$ and $\zeta = \pm 1$

2) the condition of correspondence of the points $\omega(\pm 1) = \pm l$;

3) the condition of infinity $\psi(\zeta) \to {}^1/_2\,(\sigma_y{}^\infty - \sigma_x{}^\infty)$ for $\zeta \to \infty$.

From these conditions we obtain

$$B = C = D = 0, \quad A = E = l, \quad F = {}^1/_4\,(\sigma_y{}^\infty - 5\sigma_x{}^\infty) \tag{2.4}$$

Finally, we can write solution in the form

$$\varphi(\zeta) = \frac{4\,(\sigma_x{}^\infty + \sigma_y{}^\infty)\,\zeta^2 + \sigma_y{}^\infty - 5\sigma_x{}^\infty}{16\zeta\,\sqrt{\zeta^2 - 1}} \tag{2.5}$$

$$\chi(\zeta) = l\,(1 - \zeta^2)\sqrt{\zeta^2 - 1} - l\zeta^3, \qquad \omega(\zeta) = l\,(1 - \zeta^2)\sqrt{\zeta^2 - 1} + l\zeta^3$$

The boundary of the buckled zone for all values of the load parameters $\sigma_x{}^\infty$ and $\sigma_y{}^\infty$ is an astroid $x^{2/3} + y^{2/3} = l^{2/3}$. Thus, if $\sigma_y{}^\infty < 5\sigma_x{}^\infty$ we have two different solutions to the problem: the solution of [1] and solution (2.5). It can be shown that no more one-sheeted solutions, bounded almost everywhere within the region of variation of x and y, exist. We note that the condition of one-sheetedness is essential, since it can be shown, for example, that there exists a solution (not one-sheeted, however) to the present problem in a class of functions (stresses) bounded at the ends of the segment $(-l, +l)$ and unbounded at the cusp which is the image of the point $\zeta = 0$. We require the solution to the initial problem of local buckling to be correct, i.e. to depend continuously on the parameters almost everywhere within the region of variation of the variables x and y.

In this sense solution (2.5) is incorrect, since if $\sigma_y{}^\infty = \sigma_x{}^\infty$ it does not become the solution to the corresponding elastic problem (for this value of the load parameters buckling does not occur). On the contrary, solution of [1], as can easily be verified, is correct over the whole region of its existence if $\sigma_x{}^\infty \leqslant \sigma_y{}^\infty \leqslant 5\sigma_x{}^\infty$. At the same time the solution of [1] and that given by (2.5) coincide if $\sigma_y{}^\infty = 5\sigma_x{}^\infty$. From this we may state following Theorem.

T h e o r e m 2. The solution of the initial problem exists and is unique in the class of correct, one-sheeted solutions, bounded almost everywhere; in the region of variation of parameters $\sigma_x^\infty \leqslant \sigma_y^\infty \leqslant 5\sigma_x^\infty$ this solution is given by Formulas in [1] and in the remaining region $5\sigma_x^\infty < \sigma_y^\infty$ by Formulas (2.5).

We note that the mathematical assumption that the (compressive) stresses are infinite in the neighborhood of the cusp ($\zeta = 0$) in the zone of buckling is explained physically by the fact that any membrane has some (although perhaps, extremely small) flexural rigidity which causes the high stresses necessary for the loss of stability of a very short element of the membrane in the neighborhood of the cusp.

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Translated by J.K.L.